CHAPTER 12
KINEMATICS OF A PARTICLE

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12.8 CURVILINEAR MOTION: CYLINDRICAL COMPONENTS

Polar Coordinates

Polar coordinates are particularly suitable for solving problems for which data regarding the angular motion of the radial coordinate r is given to describe the particle's motion.

Figure 1 shows the polar coordinates r and θ that specify the position of the particle P that is moving in the $xy-$plane. The origin is established at a fixed point, and the radial line r directed to the particle. The transverse coordinate θ is measured counterclockwise from a fixed reference line to the radial line.

In planar motion, the polar coordinate r is equal to the magnitude of the position vector $\vec{r}$ of the particle.

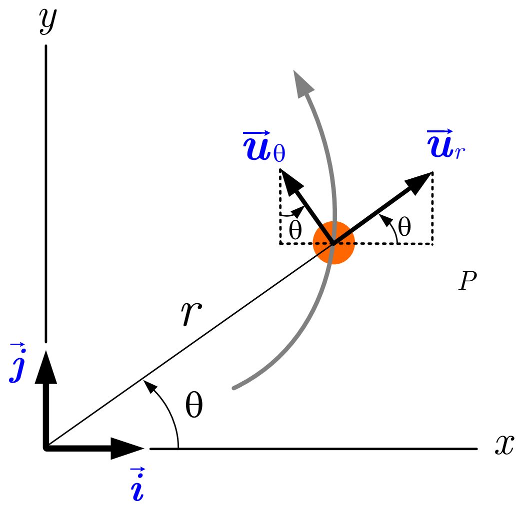


Fig. 1

Derivative of the Unit Vectors

The unit vectors $\vec{u}_r$ and $\vec{u}_\theta$ of the polar coordinate system are also shown in Fig. 1. The vector $\vec{u}_r$ is directed along the radial line, pointing away from O; $\vec{u}_\theta$ is perpendicular to $\vec{u}_r$, in the direction of increasing θ.

From Fig.1, it is clear that the unit vectors $\vec{u}_r$ and $\vec{u}_\theta$ will rotate as the particle moves. Therefore, $\vec{u}_r$ and $\vec{u}_\theta$ are the base vectors of a rotating reference frame, similar to the path $(n-t)$ coordinate system. The difference between the two coordinate systems is that $(n-t)$ coordinates depend on the path and direction of the particle, whereas polar coordinates are determined by the position of the particle. Consequently, these base vectors possess nonzero derivatives, even though their magnitudes are constant (equal to one). The time derivative of the unit base vectors can be determined by first relating the vectors to the $xy-$coordinate system. From Fig. 1,

$$\begin{aligned} \vec{u}_r &= \cos\theta\,\vec{i} + \sin\theta\,\vec{j} \\ \vec{u}_\theta &= -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} \end{aligned} \qquad (1)$$

Differentiating with respect to time while noting that $\mathrm{d}\,\vec{i}/\mathrm{d}t = \mathrm{d}\,\vec{j}/\mathrm{d}t = 0$ (the $xy-$coordinate system is fixed) yields

$$\begin{aligned} \mathrm{d}\,(\vec{u}_r)/\mathrm{d}t &= \dot{\vec{u}}_r = \dot{\theta}\left(-\sin\theta\,\vec{i} + \cos\theta\,\vec{j}\right) \\ \mathrm{d}\,(\vec{u}_\theta)/\mathrm{d}t &= \dot{\vec{u}}_\theta = \dot{\theta}\left(-\cos\theta\,\vec{i} - \sin\theta\,\vec{j}\right) \end{aligned} \qquad (2)$$

Comparing Eqs (1) and (2), one can find

$$\dot{\vec{u}}_r = \dot{\theta}\,\vec{u}_\theta, \qquad \dot{\vec{u}}_\theta = -\dot{\theta}\,\vec{u}_r \qquad (3)$$

The term $\dot{\theta}$ is called the ***angular velocity*** of the radial line. The base vectors and their derivatives are shown in Fig. 2. Notice that $\dot{\vec{u}}_r$ and $\dot{\vec{u}}_\theta$ are perpendicular to $\vec{u}_r$ and $\vec{u}_\theta$, respectively.

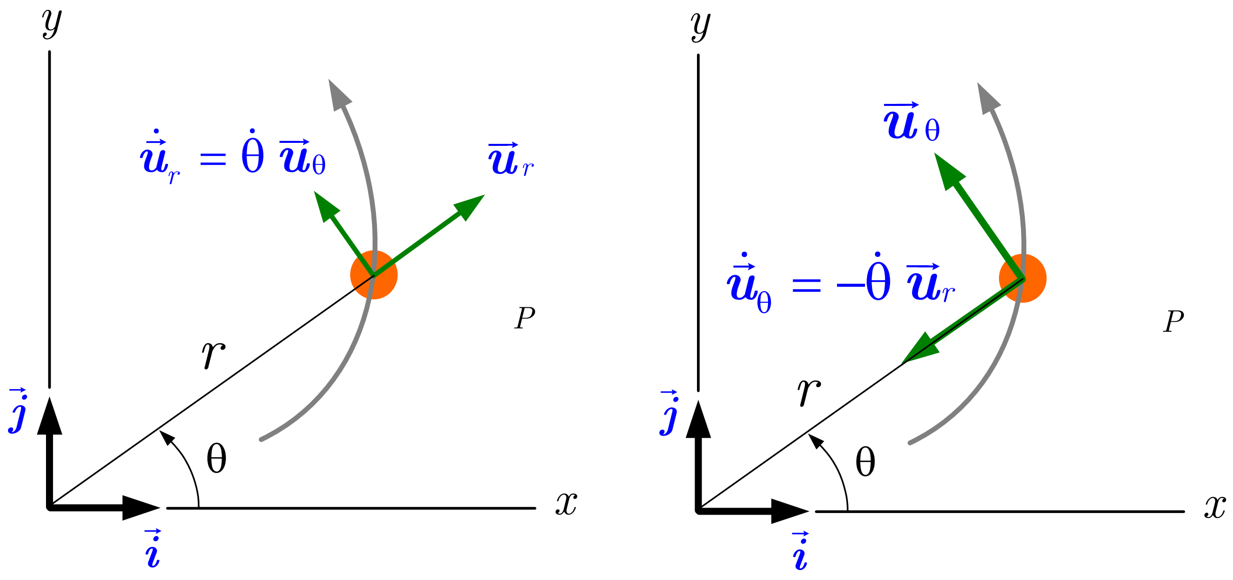


Fig. 2 Unit vectors and their derivatives

Velocity and Acceleration Vectors

The position vector $\vec{r}$ of the particle can be written in polar coordinates as

$$\vec{r} = r\,\vec{u}_r \tag{4}$$

Since the velocity vector is, by definition, $\vec{v} = \mathrm{d}\,\vec{r}/\mathrm{d}t$, we have

$$\vec{v} = \frac{\mathrm{d}\,\vec{r}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(r\,\vec{u}_r\right) = \frac{\mathrm{d}\,r}{\mathrm{d}t}\vec{u}_r + r\frac{\mathrm{d}\,\vec{u}_r}{\mathrm{d}t} = \dot{r}\,\vec{u}_r + r\,\dot{\vec{u}}_r$$

Substituting for $\mathrm{d}\left(\vec{u}_r\right)/\mathrm{d}t = \dot{\vec{u}}_r = \dot{\theta}\,\vec{u}_\theta$ from Eq.(3) gives

$$\vec{v} = v_r\,\vec{u}_r + v_\theta\,\vec{u}_\theta = \dot{r}\,\vec{u}_r + r\,\dot{\theta}\,\vec{u}_\theta \tag{5}$$

where

$$v_r = \dot{r}, \text{ and } v_\theta = r\dot{\theta} \tag{6}$$

The components v_r and v_θ are called the ***radial*** and ***transverse*** components of the velocity, respectively. These components are mutually perpendicular and the magnitude of the velocity or speed is given by

$$v = \sqrt{(\dot{r})^2 + \left(r\dot{\theta}\right)^2} \tag{7}$$

and the direction of v is always tangent to the path.

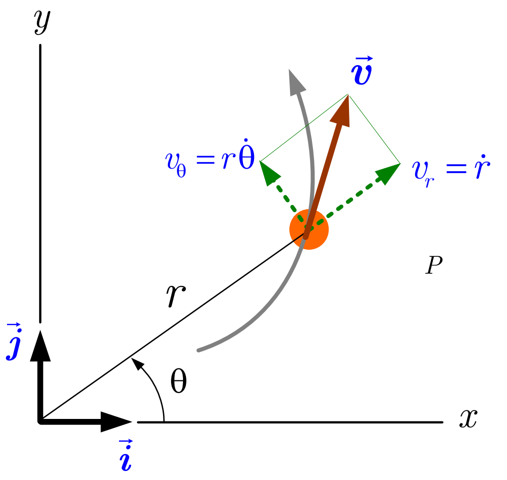


Fig. 3 Components of the velocity vector

The acceleration vector is computed as follows:

$$\begin{aligned}\vec{a} &= \frac{\mathrm{d}\,\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\,\vec{u}_r + r\dot{\theta}\,\vec{u}_\theta\right) \\ &= \left(\ddot{r}\,\vec{u}_r + \dot{r}\,\dot{\vec{u}}_r\right) + \left(\dot{r}\,\dot{\theta}\,\vec{u}_\theta + r\,\ddot{\theta}\,\vec{u}_\theta + r\,\dot{\theta}\,\dot{\vec{u}}_\theta\right)\end{aligned}$$

Substituting for $\dot{\vec{u}}_r = \dot{\theta}\,\vec{u}_\theta$ and $\dot{\vec{u}}_\theta = -\dot{\theta}\,\vec{u}_r$ from Eq.(3) gives

$$\vec{a} = \left(\ddot{r}\,\vec{u}_r + \dot{r}\,\dot{\theta}\,\vec{u}_\theta\right) + \left(\dot{r}\,\dot{\theta}\,\vec{u}_\theta + r\,\ddot{\theta}\,\vec{u}_\theta + r\,\dot{\theta}\left(-\dot{\theta}\,\vec{u}_r\right)\right)$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\vec{u}_r + \left(r\,\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right)\vec{u}_\theta = a_r\,\vec{u}_r + a_\theta\,\vec{u}_\theta$$

where

$$a_r = \left(\ddot{r} - r\dot{\theta}^2\right) \equiv \text{radial component}$$

$$a_\theta = \left(r\,\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right) \equiv \text{transverse component}$$

The term $\ddot{\theta} = \mathrm{d}^2\theta/\mathrm{d}t^2$ is called the ***angular acceleration*** since it measures the change made in angular velocity during an instant of time.

Since a_r and a_θ are always perpendicular as shown in Fig. 4, ***the magnitude*** of the acceleration is simply the positive value of

$$a = \sqrt{\left(a_r\right)^2 + \left(a_\theta\right)^2}$$

$$a = \sqrt{\left(\ddot{r} - r\dot{\theta}^2\right)^2 + \left(r\,\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right)^2}$$

The direction is determined from the vector addition of the two components. In general, a will not be tangent to the path, Fig. 4.

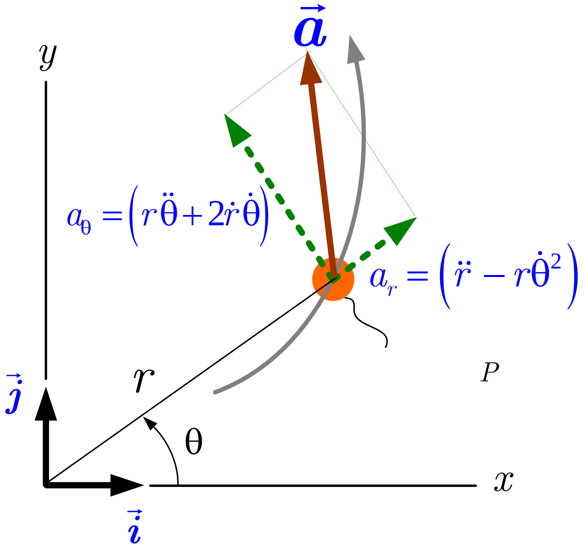


Fig. 4 Components of the acceleration vector

Cylindrical Coordinates

Motion in three-dimensions (Fig. 5) requires a simple extension of the above formulae to

Position: $\vec{r} = r\,\vec{u}_r + z\,\vec{u}_z$

Velocity: $\vec{v} = \dot{r}\,\vec{u}_r + r\dot{\theta}\,\vec{u}_\theta + \dot{z}\,\vec{u}_z$

Acceleration: $\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{u}_r + \left(r\,\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right)\vec{u}_\theta + \ddot{z}\,\vec{u}_z$

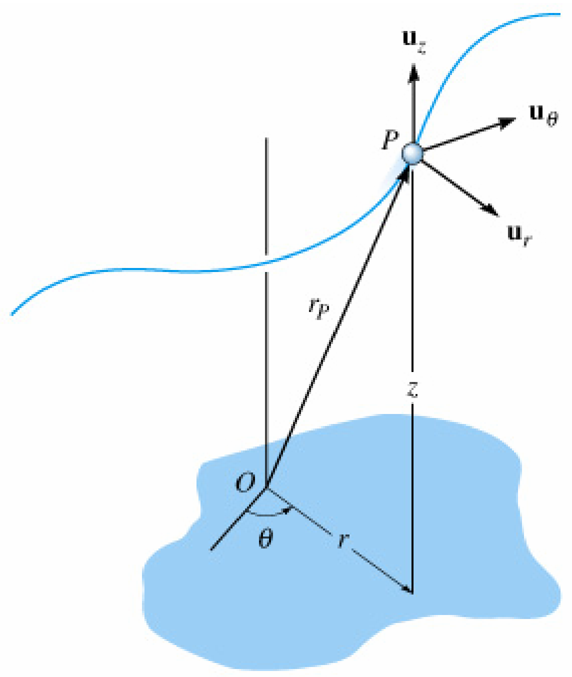
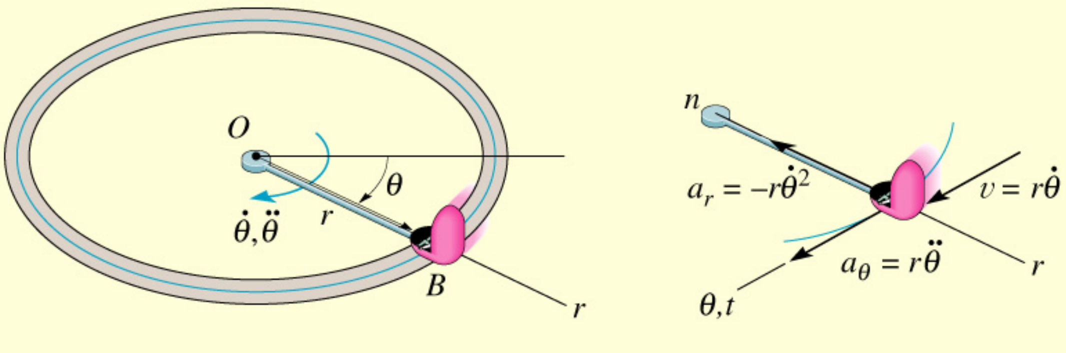


Fig. 5 Cylindrical coordinates in three dimensional motion

EXAMPLE 12.17 (TEXTBOOK)

The amusement park ride shown in Fig. 12–32*a* consists of a chair that is rotating in a horizontal circular path of radius r such that the arm OB has an angular velocity $\dot{\theta}$ and angular acceleration $\ddot{\theta}$. Determine the radial and transverse components of velocity and acceleration of the passenger. Neglect his size in the calculation.



(a) (b)

Fig. 12–32

Solution

Coordinate System. Since the angular motion of the arm is reported, polar coordinates are chosen for the solution, Fig. 12–32*a*. Here θ is not related to r, since the radius is constant for all θ.

Velocity and Acceleration. Equations 12–25 and 12–29 will be used for the solution, and so it is first necessary to specify the first and second time derivatives of r and θ. Since r is *constant*, we have

$$r = r \qquad \dot{r} = 0 \qquad \ddot{r} = 0$$

Thus

$$v_r = \dot{r} = 0 \qquad \textit{Ans.}$$

$$v_\theta = r\dot{\theta} \qquad \textit{Ans.}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 = -r\dot{\theta}^2 \qquad \textit{Ans.}$$

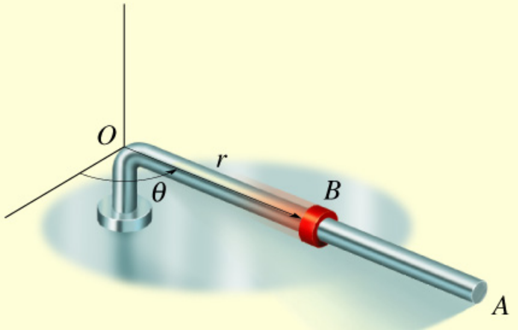
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = r\ddot{\theta} \qquad \textit{Ans.}$$

These results are shown in Fig. 12–32*b*. Also shown are the *n, t* axes, which in this special case of circular motion happen to be *colinear* with the r and θ axes, respectively. In particular note that $v = v_\theta = v_t = r\dot{\theta}$. Also,

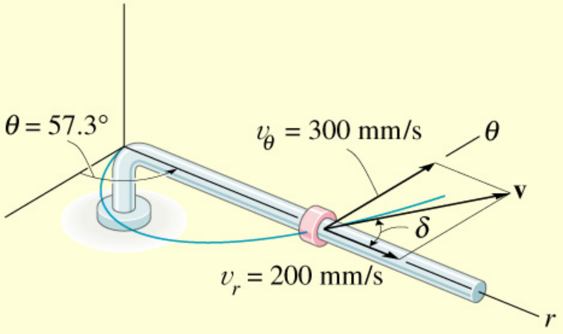
$$-a_r = a_n = \frac{v^2}{\rho} = \frac{(r\dot{\theta})^2}{r} = r\dot{\theta}^2$$

$$a_\theta = a_t = \frac{dv}{dt} = \frac{d}{dt}(r\dot{\theta}) = \frac{dr}{dt}\dot{\theta} + r\frac{d\dot{\theta}}{dt} = 0 + r\ddot{\theta}$$

EXAMPLE 12.18 (TEXTBOOK)



(a)



(b)

The rod *OA* in Fig. 12–33*a* is rotating in the horizontal plane such that $\theta = (t^3)$ rad. At the same time, the collar *B* is sliding outward along *OA* so that $r = (100t^2)$ mm. If in both cases *t* is in seconds, determine the velocity and acceleration of the collar when $t = 1$ s.

Solution

Coordinate System. Since time-parametric equations of the path are given, it is not necessary to relate *r* to *θ*.

Velocity and Acceleration. Determining the time derivatives and evaluating when $t = 1$ s, we have

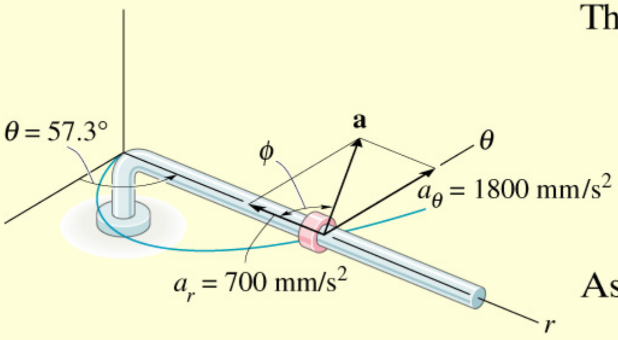
$$r = 100t^2\Big|_{t=1\text{ s}} = 100\text{ mm} \qquad \theta = t^3\Big|_{t=1\text{ s}} = 1\text{ rad} = 57.3^\circ$$

$$\dot{r} = 200t\Big|_{t=1\text{ s}} = 200\text{ mm/s} \qquad \dot{\theta} = 3t^2\Big|_{t=1\text{ s}} = 3\text{ rad/s}$$

$$\ddot{r} = 200\Big|_{t=1\text{ s}} = 200\text{ mm/s}^2 \qquad \ddot{\theta} = 6t\Big|_{t=1\text{ s}} = 6\text{ rad/s}^2.$$

As shown in Fig. 12–33*b*,

$$\begin{aligned}\mathbf{v} &= \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta \\ &= 200\mathbf{u}_r + 100(3)\mathbf{u}_\theta \\ &= \{200\mathbf{u}_r + 300\mathbf{u}_\theta\}\text{ mm/s}\end{aligned}$$



(c)

Fig. 12–33

The magnitude of **v** is

$$v = \sqrt{(200)^2 + (300)^2} = 361 \text{ mm/s} \qquad \textit{Ans.}$$

$$\delta = \tan^{-1}\left(\frac{300}{200}\right) = 56.3^\circ \qquad \delta + 57.3^\circ = 114^\circ \qquad \textit{Ans.}$$

As shown in Fig. 12–33*c*,

$$\begin{aligned}\mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta \\ &= [200 - 100(3)^2]\mathbf{u}_r + [100(6) + 2(200)3]\mathbf{u}_\theta \\ &= \{-700\mathbf{u}_r + 1800\mathbf{u}_\theta\} \text{ mm/s}^2\end{aligned}$$

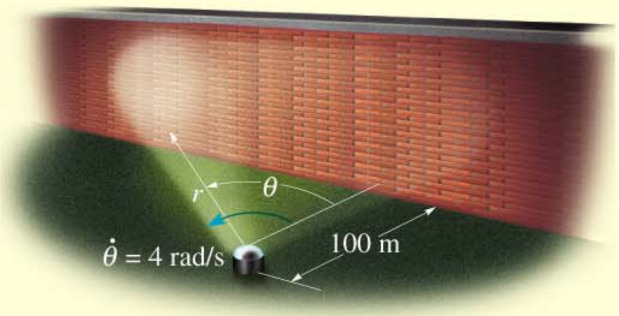
The magnitude of **a** is

$$a = \sqrt{(700)^2 + (1800)^2} = 1930 \text{ mm/s}^2 \qquad \textit{Ans.}$$

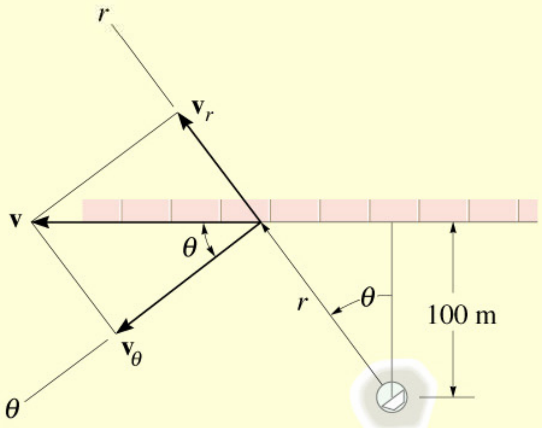
$$\phi = \tan^{-1}\left(\frac{1800}{700}\right) = 68.7^\circ \qquad (180^\circ - \phi) + 57.3^\circ = 169^\circ \qquad \textit{Ans.}$$

EXAMPLE 12.19 (TEXTBOOK)

The searchlight in Fig. 12–34*a* casts a spot of light along the face of a wall that is located 100 m from the searchlight. Determine the magnitudes of the velocity and acceleration at which the spot appears to travel across the wall at the instant $\theta = 45°$. The searchlight is rotating at a constant rate of $\dot{\theta} = 4$ rad/s.



(a)



(b)

Solution

Coordinate System. Polar coordinates will be used to solve this problem since the angular rate of the searchlight is given. To find the necessary time derivatives it is first necessary to relate r to θ. From Fig. 12–34*a*, this relation is

$$r = 100/\cos\theta = 100\sec\theta$$

Velocity and Acceleration. Using the chain rule of calculus, noting that $d(\sec\theta) = \sec\theta\tan\theta\, d\theta$, and $d(\tan\theta) = \sec^2\theta\, d\theta$, we have

$$\dot{r} = 100(\sec\theta\tan\theta)\dot{\theta}$$

$$\begin{aligned}\ddot{r} &= 100(\sec\theta\tan\theta)\dot{\theta}(\tan\theta)\dot{\theta} + 100\sec\theta(\sec^2\theta)\dot{\theta}(\dot{\theta}) \\ &\quad + 100\sec\theta\tan\theta(\ddot{\theta}) \\ &= 100\sec\theta\tan^2\theta(\dot{\theta})^2 + 100\sec^3\theta(\dot{\theta})^2 + 100(\sec\theta\tan\theta)\ddot{\theta}\end{aligned}$$

Since $\dot{\theta} = 4$ rad/s = constant, then $\ddot{\theta} = 0$, and the above equations, when $\theta = 45^\circ$, become

$$r = 100 \sec 45^\circ = 141.4$$

$$\dot{r} = 400 \sec 45^\circ \tan 45^\circ = 565.7$$

$$\ddot{r} = 1600(\sec 45^\circ \tan^2 45^\circ + \sec^3 45^\circ) = 6788.2$$

As shown in Fig. 12–34*b*,

$$\begin{aligned} \mathbf{v} &= \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta \\ &= 565.7\mathbf{u}_r + 141.4(4)\mathbf{u}_\theta \\ &= \{565.7\mathbf{u}_r + 565.7\mathbf{u}_\theta\} \text{ m/s} \\ v &= \sqrt{v_r^2 + v_\theta^2} = \sqrt{(565.7)^2 + (565.7)^2} \\ &= 800 \text{ m/s} \end{aligned}$$

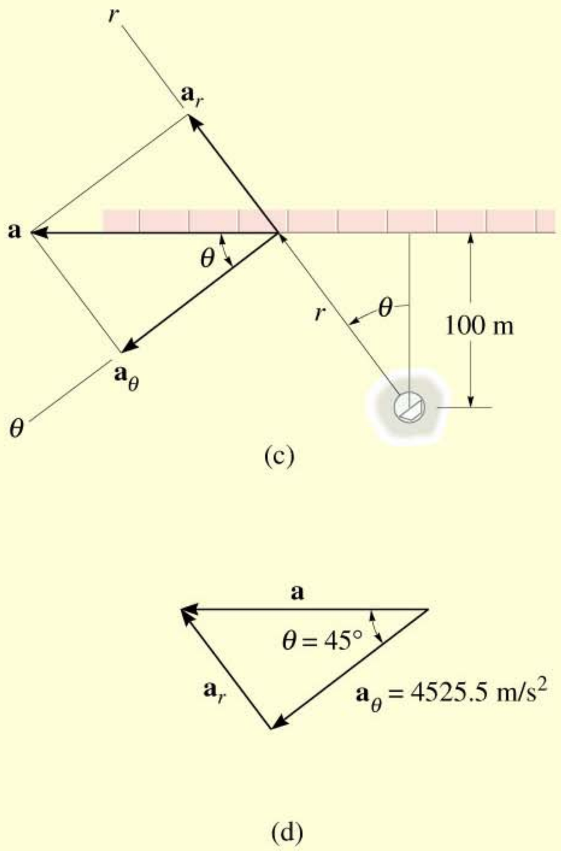
Ans.

As shown in Fig. 12–34*c*,

$$\begin{aligned} \mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta \\ &= [6788.2 - 141.4(4)^2]\mathbf{u}_r + [141.4(0) + 2(565.7)4]\mathbf{u}_\theta \\ &= \{4525.5\mathbf{u}_r + 4525.5\mathbf{u}_\theta\} \text{ m/s}^2 \\ a &= \sqrt{a_r^2 + a_\theta^2} = \sqrt{(4525.5)^2 + (4525.5)^2} \\ &= 6400 \text{ m/s}^2 \end{aligned}$$

Ans.

Note: It is also possible to find a without having to calculate $\ddot{r}$ (or a_r). As shown in Fig. 12–34*d*, since $a_\theta = 4525.5$ m/s^2, then by vector resolution, $a = 4525.5/\cos 45^\circ = 6400$ m/s^2.



(c)

(d)

Fig. 12–34